Lecture5:

Generalization of the Weierstrass Theorem

1. Generalization of the Weierstrass Theorem
2. Convergence problem
3. Theorem 1
4. Relaxing the conditions
5. Theorem 2
6. Example
7.

(1) Generalization of the Weierstrass Theorem

- Let $f(x^1, x^2, \ldots, x^n)$ be a continuous in the domain

$$a_i \le x_i \le b_i \qquad (i = 1, 2, \ldots, n)$$

This function can be approximated uniformly with any desired accuracy as a linear combination of the monomials

$$x_1^{m_1} \, x_2^{m_2} \ldots x_n^{m_n} \quad , \quad m_i \geq 0$$

i.e

$$f^M = \sum_{m_1, \ldots, m_n \le M}' A_{m_1 \ldots m_n} \, x^{m_1} x_2^{m_2} \ldots x^{m_n}$$

$$|f - f^M| < \varepsilon \qquad \text{(approximation)}$$

- Consider functions of two variables $f(x,y)$
let $-1 < x, y < 1$

$$x = r\cos\theta \, , \quad y = r\sin\theta$$

$$f^M(x,y) = \sum_{i,j \le M} a^M_{ij} \, x^{m_i} y^{m_j}$$

$$= \sum_{m_i \le M}' b^M_{ij} \cos^{m_i}\theta \, \sin^{m_j}\theta$$

Let $f(x,y) = r f(\theta)$, $\displaystyle f_n(\theta) = \sum^{n} a_k^{(n)} e^{ik\theta}$

$$|f(\theta) - f_{(n)}(\theta)| < \varepsilon$$

given ε there exists a function (sequence of functions) $f_{(n)}$ and an integer n so that

$$|f(\theta) - f_{(n)}(\theta)| < \varepsilon \qquad \theta \in [-\pi, \pi]$$

provided that $f(\theta)$ is continuous and satisfies the periodicity condition

$$f(\pi) = f(-\pi).$$

. Basis: The continuous functions

$$\frac{1}{\sqrt{2\pi}} e^{im\theta} \quad , \quad m = 0, \pm 1, \pm 2, \ldots$$

form a basis of $L_1^2(-\pi, \pi)$

$$|e_m\rangle = \frac{1}{\sqrt{2\pi}} e^{im\theta} \quad ,$$

$$\langle e_m | e_n \rangle = \delta_{mn}$$

$$|e_0^+\rangle = |e_0\rangle = \frac{1}{\sqrt{2\pi}}$$

$$|e_m^+\rangle = \frac{1}{\sqrt{2}} [|e_m\rangle + |e_{-m}\rangle] \to \frac{1}{\sqrt{2\pi}} \cos m\theta$$

$$|e_m^-\rangle = \frac{1}{i\sqrt{2}} [|e_m\rangle - |e_{-m}\rangle] \to \frac{1}{\sqrt{2\pi}} \sin m\theta$$

② The convergence problem:

Any vector $|f\rangle \in L_1^2(-\pi,\pi)$ is a limit vector of a series of vectors

$$|f_n\rangle = \sum_{k=-n}^{n} a_k^{(n)} |e_k\rangle$$

In the sense that

$$\rho^2(|f\rangle, |f_n\rangle) = \left\{ \langle f|f\rangle - \sum_{k=-n}^{n}{}' |\langle e_k|f\rangle|^2 \right\}$$

$$+ \sum_{k=-n}^{n}{}' |a_{(k)}^{(n)} - \langle e_k|f\rangle|^2$$

$$\rho \xrightarrow{n\to\infty} 0 \quad \Rightarrow \quad a_{(k)}^{(n)} = \langle e_k|f\rangle \qquad n\to\infty$$

Hence the function

$$\sum_{k=-n}^{n}{}' \langle e_k|f\rangle e_{(k)}(x)$$

is the best approximation

$$|f\rangle = \lim_{n\to\infty} |f_n\rangle = \lim_{n\to\infty} \sum_{k=-n}^{n} a_k^{(n)} |e_k\rangle$$

$$\overset{?}{=} \sum_{k=-\infty}^{\infty} \langle e_k|f\rangle |e_k\rangle$$

uniform convergence

We have two facts

i) $\rho \to 0$ means

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-n}^{n} \langle e_m | f \rangle e_m \right|^2 d\theta = 0 \qquad *$$

ii) Generalize Weierstrass theorem: Any continuous function $f(\theta)$ that satisfies the periodicity condition $f(\pi) = f(-\pi)$ can be approximated uniformly by linear combination of $\frac{1}{\sqrt{2\pi}} e^{im\theta}$

$$\left| f(\theta) - \sum_{m=-n}^{n} a_m^{(n)} \frac{1}{\sqrt{2\pi}} e^{im\theta} \right| \xrightarrow[n\to\infty]{} 0$$

However, (inspite of $a_n^{(n)} = f^m$ as $n \to \infty$)
It is not necessarily true that the expansion

$$f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \langle e_m | f \rangle e^{im\theta} \quad \text{or} \quad \text{[illegible]}$$

RHS converges to f since it may happen that

$$\lim_{n\to\infty} \sum_{m=-n}^{n} a_m^{(n)} \frac{1}{\sqrt{2\pi}} e^{im\theta} \neq \sum_{m=-\infty}^{\infty} \lim_{n\to\infty} a_m^{(n)} \frac{1}{\sqrt{2\pi}} e^{im\theta}$$

for a set of arguments θ of zero total measure, this would not contradict (*)

One has to add additional conditions on a function to ensure the convergence of its Fourier series

$$\lim |f^n\rangle = \lim_{n\to\infty} \sum a_n^{(n)} \frac{e^{in\theta}}{\sqrt{2\pi}}$$

Theorem 1. Let a function $f(\theta)$ and its derivatives be continuous for $-\pi \le \theta \le \pi$ and satisfy the periodicity condition

$$f(\pi) = f(-\pi)$$

Then the Fourier series

$$\frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} f^{m}_{(\infty)} e^{im\theta}$$

with

$$f^{m} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(\theta) e^{-im\theta} d\theta = \langle e_m | f \rangle$$

converges uniformly to $f(\theta)$ in the interval $[-\pi, \pi]$.

Proof: $\frac{df}{d\theta} = f' \in L^2(-\pi, \pi)$.

$$\frac{df}{d\theta} = \sum_{-\infty}^{\infty} \langle e_m | f' \rangle | e_m \rangle$$

$$\langle e_m | f' \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{df}{d\theta} e^{-im\theta} d\theta$$

$$= \frac{1}{\sqrt{2\pi}} \left[f(\pi)(-1)^m - f(-\pi)(-1)^m + im \int f e^{-im\theta} \right.$$

$$= im f^{m}$$

$$\left|\sum_{k\le|m|\le l} f^m e^{im\theta}\right|^2 \le \sum_{k\le|m|\le l} |f^m|^2 \le \sqrt{\sum_{k\le|m|\le l} |mf^m|^2}\,\sqrt{\sum_{|m|} \frac{1}{m^2}}$$ 65 74

$$\langle f'|f'\rangle = \sum_m |\langle e_m|f'\rangle|^2 = \sum_{-\infty}^{\infty} m^2 |f^m|^2 < \infty$$

$$\Rightarrow \langle f|f\rangle = \sum |f^m|^2$$

According to Cauchy-Schwarz Inequality

$$\left|\sum f^m e^{im\theta}\right|^2 \le \sum |f^m|^2$$

$$\left|\sum^{\infty} f^m e^{im\theta}\right|^2 < \infty$$

This sum converges

$$\le \sum |m f^m|^2 \sum_{|m|} \frac{1}{m^2} < \infty$$

$$\Rightarrow f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty}{}' f^m e^{im\theta}$$

convergence is unif.

$$\sum_{m=-\infty}^{\infty} f^m |e_m\rangle = f$$

$$f^m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-im\theta} f(\theta)\, d\theta$$

- In the theorem we did not use the continuity of the first derivative of f. Hence it can be weakened. This theorem is valid for the case where $df/d\theta$ makes a finite number of jumps.

- The condition of periodicity is abandoned, f' satisfies the above condition, has only finite number of jumps and $f(\theta)$ has only discontinuities of the first kind ($f(\theta_0 \pm 0) = \lim_{\epsilon\to 0} f(\theta_0 \pm \epsilon)$ but $f(\theta_0+0) \ne f(\theta_0-0)$).

$\Rightarrow$

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \sum_{m=-n}^{n}{}' f^m e^{im\theta} = \begin{cases} \frac{1}{2}[f(\theta+0)+f(\theta-0)], & -\pi < \theta < \pi \\ \frac{1}{2}[f(\pi)+f(-\pi)]. \end{cases}$$

The convergence is uniform in every closed interval where $f(\theta)$ is continuous. Proof consist of expressing $f(\theta)$ as a sum of two functions, one of which is continuous and the other discontinuous.

- This theorem is generalized to more general functions called "bounded variation"

A function is called "to be of bounded variation" on an interval $[a,b]$ if it can be written as the sum of two function $f(\theta) = f_1(\theta) + f_2(\theta)$ where one of them is nonincreasing and bounded and the other is nondecreasing and bounded ($f(\theta+\varepsilon)$ and $f(\theta_0-\varepsilon)$ exist)

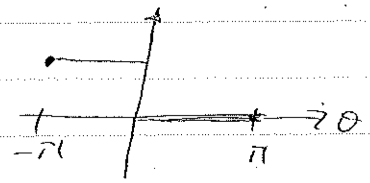
Theorem: The Fourier series of a function $f(\theta)$ that is of bounded variation for $-\pi \le \theta \le \pi$ converges to

$$\frac{1}{2}\left[f(\theta+0) + \frac{1}{2} f(\theta-0)\right] \quad \text{for} \quad -\pi < \theta < \pi$$

$$\frac{1}{2}\left[f(\pi) + f(-\pi)\right] \quad \text{for} \quad \theta = \pm\pi$$

moreover, in every closed interval where $f(\theta)$ is continuous, the convergence is uniform

Example: $f(\theta) = \begin{cases} 0 & 0 < \theta \le \pi \\ 1 & -\pi \le \theta < 0 \end{cases}$



$$f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} f^m e^{im\theta} \quad , \quad f^m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(\theta) e^{-im\theta} d\theta$$

$$f^m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{0} e^{-im\theta} d\theta = \frac{i}{m\sqrt{2\pi}} \left[1-(-1)^m\right], \quad m \neq 0$$

$$f^0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{0} d\theta = \sqrt{\frac{\pi}{2}}$$

$$f(\theta) = \frac{1}{2} + \frac{i}{2\pi} \sum_{\substack{-\infty \\ m\neq 0}}^{\infty} \frac{(1-(-1)^m)}{m} e^{im\theta} = \frac{1}{2} - \frac{2}{\pi} \sum_{\substack{m>0 \\ \text{odd}}} \frac{\sin m\theta}{m}$$

at $\theta = 0$ (discont. of f) $f(0) = 1/2$ $\left(= \frac{1}{2}\left(f(+0) + f(-0)\right)\right)$

for an arbitrary interval $[-l, l]$

$$f(\theta) = \frac{1}{\sqrt{2l}} \sum_{-\infty}^{\infty} f^m e^{im(\pi/l)\theta}$$

$$f^m = \frac{1}{\sqrt{2l}} \int_{-l}^{l} f(\theta) e^{-im(\pi/l)\theta} d\theta$$

either f is defined as $[-l, l]$ or function that periodic with period $2l$

$$f(\theta + 2l) = f(\theta).$$

1) $f(\theta)$ is even

$$f(\theta) = a^0 + \sum_{m=1}^{\infty} a_m \cos m(\pi/l)\theta.$$

$$a_0 = \frac{1}{l}\int_0^l f(\theta) d\theta, \quad a_m = \frac{2}{l}\int_0^l f(\theta) \cos m(\pi/l)\theta \, d\theta \quad m>0$$

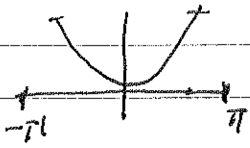
2) $f(\theta)$ is odd

$$f(\theta) = \sum_{m=1}^{\infty} b_m \sin m(\pi/l)\theta$$

$$b_m = \frac{2}{l}\int_0^l f(\theta) \sin m(\pi/l)\theta$$

Example : $f(\theta) = \theta^2 \quad 0 < \theta < \pi$

f(θ) is an even function



It admits the expansion for even funct.

$$a^0 = \frac{1}{\pi}\int_0^\pi \theta^2\, d\theta = \frac{\pi^2}{3}.$$

$$a^m = \frac{2}{\pi}\int_0^\pi \theta^2 \cos m\theta\, d\theta = \frac{4(-1)^m}{m^2}$$

$$\Rightarrow \theta^2 = \frac{\pi^2}{3} + 4\sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \cos m\theta$$

$\theta = 0$

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + (-1)^{m-1}\frac{1}{m^2} + .$$

$\theta = \pi$ $$\pi^2 - \frac{\pi^2}{3} = 4\sum_{m=1}^{\infty} \frac{1}{m^2}$$

$$\sum_1^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$$

Lecture 6: Generalized functions:

1. An example: $D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2}$
2. Definitions: good, fairly good, generalized function
3. properties
4. Delta function
5. Theta functions
6. Fourier transform of GF

$$\int_{-\infty}^{\infty} e^{2ax+bx^2}\, dx = \sqrt{\frac{\pi}{b}}\, e^{-(a^2/\cancel{4}b)} \qquad b \neq 0$$

Generalized Functions:

Consider the functions

$$D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \qquad n = 1, 2, \dots$$

as $n \to \infty$, $D_n(x)$ behaves

$$\lim_{n\to\infty} D_n(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

On the other hand, since

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\, dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{-y^2}\, dy = \frac{1}{a}\sqrt{\pi}, \qquad a > 0$$

then,

$$\int_{-\infty}^{\infty} D_n(x)\, dx = \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} dx = \sqrt{\frac{n}{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{n}} = 1$$

for all n. For any smooth enough function $f(x)$, the integral

$$\int_{-\infty}^{\infty} D_n(x) f(x)\, dx \xrightarrow{n\to\infty} f(0)$$

formally on writes that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} D_n(x)\, f(x)\, dx = \int_{-\infty}^{\infty} \delta(x)\, f(x) = f(0)$$

Remark since the integrals are not uniformly convergent

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \neq \int_{-\infty}^{\infty} \lim_{n\to\infty}$$

Definition of Generalized Function

Definition 1 : A function $f(x)$ is called "a good function" if it is differentiable everywhere any number of times and it and all its derivatives vanish as $|x| \to \infty$ faster than any power of $\frac{1}{|x|}$

Definition 2 : A function $f(x)$ is called "a fairly good function" if it is differentiable everywhere any number of times and its modulus and that of it derivatives ~~vanish~~ doesnot increase faster than some power of $|x|$ as $|x| \to \infty$

Examples

(a) Fairly good functions: Any polynomial of x

(b) Good functions : $D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2}$

. differentiable everywhere

. $\frac{d^m}{dx^m} D_n(x) = \frac{(-1)^m}{n^{m/2}} H_m(\sqrt{n}\; x)\; D_n(x)$

Where H_m is the Hermite polynomial

. as $|x| \to \infty$ $\frac{d^m}{dx^m} D_n(x)$ vanish

faster than any power of $\frac{1}{|x|}$.
Because of the existence of e^{-nx^2}

Definition 3 Let $h_n(x)$ be good functions.
A generalized function $\chi(x)$ is a sequence of functions $h_n(x)$ such that for any good function $g(x)$, the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} \chi(x)\, g(x)\, dx$$

exists. and equals to

$$\int_{-\infty}^{\infty} \chi(x)\, g(x)\, dx$$

Remarks

a) Two generalized functions $\alpha(x)$ and $\beta(x)$ are considered to be equal if the corresponding sequences satisfy

$$\text{(i)} \quad \lim_{n\to\infty} \int_{-\infty}^{\infty} \alpha_n(x)\, g(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \beta_n(x)\, g(x)\, dx$$

for any good function $g(x)$. Any two sequences satisfying ~~the same definition~~ (*) define the same distribution

b) (i) a sequence of "good function" $f(x)$

$h_n(x) = \{ f(x), f(x), \ldots, f(x), \ldots \}$

this defines, trivially a distribution

(ii) Let $f(x)$ be a "fairly good function" Then each $f_n(x)$ defined by

$$f_n(x) = e^{-x^2/n} f(x) \qquad n = 1, 2, \ldots$$

is a good function. This sequence defines a distribution

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

Propechies of GF

(a) addition of GF

let $a_n(x)$ and $b_n(x)$ define the distribut (GF) $\alpha(x)$ and $\beta(x)$. The generalized function $\chi(x)$ defined by the sequence $h_n(x) = a_n(x) + b_n(x)$ ~~$\chi(x)$~~ is called the sum of $\alpha(x)$ and $\beta(x)$.

proof:

$$\int_{-\infty}^{\infty} \chi(x)\, g(x)\, dx = \lim_{n\to\infty} \int_{-\alpha}^{\alpha} h_n(x)\, g(x)\, dx$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} [a_n(x) + b_n(x)]\, g(x)\, dx$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} a_n(x)\, g(x)\, dx + \lim_{n\to\infty} \int_{-\infty}^{\infty} b_n(x)\, g(x)\, dx$$

$$= \int_{-\infty}^{\infty} \alpha(x)\, g(x)\, dx + \int_{-\infty}^{\infty} \beta(x)\, g(x)\, dx$$

$$= \int_{-\infty}^{\infty} (\alpha(x) + \beta(x))\, g(x)\, dx$$

b) multiplication of GFs.

Let the sequence $a_n(x)$ define the distribution $\alpha(x)$ and $f(x)$ be a fairly good function. The generalized function $\chi(x)$ defined by the sequence

$$h_n(x) = f(x)\, a_n(x)$$

is called the product of f and $\alpha(x)$

$$\chi(x) = f(x)\, \alpha(x)$$

good × good = good

fairly good × good = good

fairly good × fairly good = fairly good

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fairly good times a distribution and good function time a distribution are possible products BUT products of two GF is not defined in general.

The problem is the following.

let $a_n \to \alpha(x)$, $b_n \to \beta(x)$

or $\int_{-\infty}^{\infty} a_n(x)\, g(x)\, dx$, and $\int_{-\infty}^{\infty} b_n(x)\, g(x)\, dx$

convergence of these sequence does not imply the convergence of the product

$$\int_{-\infty}^{\infty} a_n(x)\, b_n(x)\, g(x)$$

in general.

c) Differentiation of the GFs.

let $h_n(x) = \frac{d}{dx} a_n(x)$

$$\int_{-\infty}^{\infty} h_n(x)\, g(x)\, dx = \int \frac{d}{dx} a_n \; g(x)$$

$$= + \underbrace{a_n(x)\, g(x) \Big|_{-\infty}^{\infty}}_{=0} - \int a_n \frac{dg}{dx}\, dx$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(x)\, g(x)\, dx = - \int \alpha(x) \frac{dg}{dx}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{d\alpha}{dx}\, g\, dx$$

hence $\alpha_n \to \alpha$

$$\frac{d\alpha_n}{dx} \to \frac{d\alpha}{dx}$$

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prove that

a) $\frac{d}{dx}\left(\alpha(x)+\beta(x)\right) = \frac{d\alpha}{dx} + \frac{d\beta}{dx}$

b) $\frac{d}{dx}(f\alpha) = \frac{df}{dx}\alpha + f\frac{d\alpha}{dx}$

c) $\frac{d}{dx}\alpha(ax+b) = a\frac{d\alpha}{dy}\Big|_{y=ax+b}$

Dirac δ Function $\delta(x)$

consider the sequence

$$D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \qquad n = 1, 2, \ldots$$

remember that

$$\lim_{n\to\infty} D_n(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

i) $$\int_{-\infty}^{\infty} D_n(x)\, dx = 1$$

or $$\int_{-\infty}^{\infty} e^{-nx^2}\, dx = \sqrt{\frac{\pi}{n}} \qquad n = 1, 2, \ldots$$

ii) $$\lim_{n\to\infty} \int_{-\infty}^{\infty} D_n(x)\, f(x)\, dx = f(0)$$

proof: $$\int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \left(f(x) - f(0)\right)$$

$$+ \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} f(0)\, dx$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \left(f(x) - f(0)\right) dx + f(0)$$

$$\frac{f(x)-f(0)}{x} = \frac{df}{dx}\Big|(\xi), \quad 0 < \xi < x$$

$$\left| \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \left[f(x) - f(0) \right] dx \right|$$

$$\leq \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \left| x \left(\frac{df}{dx}\right)_{\xi} \right| dx$$

$$\leq \left| \left(\max \frac{df}{dx} \right) \right| \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nx^2} |x| dx$$

$$\int_{-\infty}^{\infty} e^{-nx^2} |x| dx = 2 \int_{0}^{\infty} x e^{-nx^2} dx$$

$$= \frac{1}{n} \int_{0}^{\infty} e^{-nx^2} d(nx^2) = \frac{\sqrt{\pi}}{2n}$$

$$\left| \int_{-\infty}^{\infty} \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \left[f(x) - f(0) \right] dx \right| \leq \cancel{\tfrac{1}{2}} \left| \max \left(\frac{df}{dx} \right) \right| \frac{1}{\sqrt{n}}$$

hence

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} D_n(x)\, f(x)\, dx = f(0)$$

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$$a_n(x) = \frac{1-\cos nx}{\pi n x^2}$$

$$\int_{-\infty}^{\infty} \frac{1-\cos nx}{\pi n x^2} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1-\cos nx}{\pi n x^2} \cdot [f(x) - f(0)]\,dx$$

$$+ \int_{-\infty}^{\infty} \frac{1-\cos nx}{\pi n x^2}\,dx \quad f(0)$$

$$\frac{1-\cos nx}{x^2} = \frac{1-(\cancel{1}-2\sin^2\frac{nx}{2})}{x^2} = 2\sin^2\frac{nx}{2} \,/ x^2$$

$$\int \frac{1-\cos nx}{\pi n x^2}\,dx = \int \frac{2}{\pi n}\,\frac{2}{n}\,\frac{\sin^2\frac{nx}{2}}{(\frac{n}{2}x)^2\,\frac{4}{n^2}}\,d\left(\frac{xn}{2}\right) = \frac{\cancel{n}}{\cancel{2}\pi n}\int \frac{\sin^2 y}{y^2}\,dy$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2}\,dy = 1.$$

$$\int_{-\infty}^{\infty} \frac{1-\cos nx}{\pi n x^2} f(x)\,dx = f(0) + \int_{-\infty}^{\infty} \frac{1-\cos nx}{\pi n x^2}\,[f(x)-f(0)]\,dx$$

$$\left|\int_{-\infty}^{\infty} \frac{1-\cos nx}{\pi n x^2}\,(f(x)-f(0))\,dx\right| \le \int_{-\infty}^{\infty} \frac{1-\cos nx}{\pi n x^2}\, x \left|\frac{df}{dx}\right| dx$$

$$\le \max\left|\frac{df}{dx}\right| \frac{1}{\pi n} \int_{-\infty}^{\infty} \frac{1-\cos nx}{|x|}\,dx$$

$$\le \max\left|\frac{df}{dx}\right| \frac{1}{\pi n}\, 2 \int_{-\infty}^{\infty} \frac{\sin^2\frac{nx}{2}}{|x|}\,dx$$

$$\le \max\left|\frac{df}{dx}\right| \frac{4}{\pi n} \int_0^{\infty} \frac{\sin y}{y}\,dy \longrightarrow 0$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} a_n(x)\, f(x)\, dx = \int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx$$

$$= f(0)$$

(91)

(yep)

$$f_n(x) = \begin{cases} 0 & -\infty < x \le -\frac{1}{n} \\ \frac{1}{2}(nx+1) & -\frac{1}{n} \le x \le \frac{1}{n} \\ 1 & \frac{1}{n} \le x < \infty \end{cases}$$

$$\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f_n(x)[g(x) - \cancel{g(0)]\,dx} +$$

$$= \int_{-\infty}^{-\frac{1}{n}} 0 \quad g + \int_{-1/n}^{1/n} \frac{1}{2}(nx+1)\, g(x)\, dx$$

$$+ \int_{1/n}^{\infty} 1 \; g(x)\, dx$$

$$= \frac{1}{2} n \int_{-1/n}^{1/n} x\, g(x)\, dx + \frac{1}{2} \int_{-1/n}^{1/n} g(x)\, dx$$

$$+ \int_{1/n}^{\infty} g(x)\, dx$$

$$\left| \frac{1}{2} n \int_{-1/n}^{1/n} x\, g(x)\, dx + \frac{1}{2} \int_{-1/n}^{1/n} g(x)\, dx \right| \le \frac{1}{2} n \max g \int_{-1/n}^{1/n} |x|\, dx$$

$$+ \frac{1}{2} \max|g| \frac{1}{n} 2$$

$$\le \max|g| \left(\frac{n}{2}\, 2\, \frac{1}{n^2} + \frac{1}{n} \right) = \frac{2}{n} \max|g|$$

$$\to 0$$

$\Rightarrow$

1(b) $$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,g(x)\,dx = \int_0^{\infty} g(x)\,dx$$

$$= \int_{-\infty}^{\infty} \theta(x)\,g(x)$$

$\to$ step function

(yes)

$$\frac{df_n}{dx} = \begin{cases} 0 & -\infty < x < -\frac{1}{n} \\ \frac{n}{2} & -\frac{1}{n} < x < \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{df_n}{dx}\,g(x) = \int_{-1/n}^{1/n} \frac{n}{2}\,g(x)\,dx = \frac{n}{2}\int_{-1/n}^{1/n} g(x)\,dx$$

$$= \cancel{\frac{n}{2}\int_{-\infty}^{\infty} g(x)}$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\left(\frac{df_n}{dx}\right)g(x)\,dx = \frac{\lim}{2}\,\frac{1}{1/n}\int_{-1/n}^{1/n} g(x)\,dx$$

$$= \frac{1}{2}\lim_{u\to 0}\frac{1}{u}\int_{-u}^{u} g(x)\,dx$$

$$= \frac{1}{2}\,2\,g(0) = g(0)$$

$$= \int \delta(x)\,g(x)\,dx$$

Some further properties

(1) $D_n(ax)$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} D_n(ax)\, g(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\frac{n}{\pi}}\, e^{-nx^2a^2} g(x)dx$$

$$= \int_{-\infty}^{\infty}\delta(ax)\, g(x) = \lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\frac{n}{\pi}}\, e^{-nu^2}\, g\left(\frac{u}{|a|}\right)\frac{du}{|a|}$$

$|a|x = u$

$$= \frac{1}{|a|}\, g(0) = \int_{-\infty}^{\infty}\frac{1}{|a|}\,\delta(x)\, g(x)\,dx$$

$$\Rightarrow \quad \delta(ax) = \frac{1}{|a|}\,\delta(x)$$

(2) $D_n(f(x))$

$f(x)$ has one single root x_0 $(f(x_0)=0)$

let $f(x) = (x-x_0)\, h(x)$

$h(x)$ ~~is single~~ doesn't change sign.

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\frac{n}{\pi}}\, e^{-n(x-x_0)^2h^2} g(x)\,dx = \int \delta(f)\, g(x)\,dx$$

let $(x-x_0)|h| = u$, $\quad (|h| + (x-x_0)|h|')\,dx = du$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\sqrt{\frac{n}{\pi}}\, e^{-nu^2}\frac{g(x(u))\,du}{\frac{du}{dx}} = \left.\frac{g(x(u))}{\frac{du}{dx}}\right|_{u=0}$$

$$= \frac{g(x_0)}{|h|_{x_0}} = \frac{g(x_0)}{\left|\frac{du}{dx}\right|_{x_0}} \Rightarrow \delta(f) = \frac{\delta(x-x_0)}{\left|\frac{df}{dx}\right|_{x_0}}$$

$$\frac{du}{dx} = |h|$$

(3) ~~$D\delta(f(x))$ ... N-zeros~~

$D_n(f(x))$ f has N-zeros

$$\delta(f(x)) = \sum_{i=1}^{N} \frac{\delta(x-x_i)}{\left|\left.\frac{df}{dx}\right|_{x=x_i}\right|}$$

$$\delta(x^2-a^2) = \frac{1}{2|a|}\left(\delta(x-a)+\delta(x+a)\right)$$

(4)

$$D_n(x-a) \longrightarrow \delta(x-a)$$

$$\lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} D_n(x-a)\, g(x)\, dx$$

$$x-a = \frac{u}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} \sqrt{\frac{n}{\pi}}\, e^{-n(x-a)^2} g(x)\, dx$$

$$\sqrt{\frac{1}{\pi}}\, e^{-u^2}$$

$$= \lim_{n\to\infty} \int_{-\sqrt{n}(\varepsilon+a)}^{\sqrt{n}(\varepsilon-a)} \sqrt{\frac{n}{\pi}}\, e^{-nu^2} g\left(a+\frac{u}{\sqrt{n}}\right) du$$

$= g(a)$ $|a| < \varepsilon$

$= 0$ $|a| > \varepsilon$

This implies $\displaystyle\int_{-\varepsilon}^{\varepsilon} \delta(x-a)\, g(x)\, dx = \begin{cases} g(a) & |a| \le \varepsilon \\ 0 & |a| > \varepsilon \end{cases}$

5 $f(x)\,\delta(x) = f(0)\,\delta(x)$

6 $$h_n(x) = \omega \sum_{m=0}^{n} u_m(x)\, u_m(x_0)$$

$$\int_a^b h_n(x)\, g(x)\, dx = \sum u_m(x_0) \int_a^b \omega u_m(x)\, g(x)\, dx$$

$$g_m = \int_a^b \omega\, u_m(x)\, g(x)\, dx$$ "gen. Fourier Coef"

$$\int_a^b h_n(x)\, g(x)\, dx = \sum_{m=0}^{n} u_m(x_0)\, g_m$$

$$\lim_{n\to\infty} \int_a^b h_n(x)\, g(x)\, dx = \sum_{m=0}^{\infty} g_m\, u_m(x_0) = g(x_0)$$

$$\Rightarrow \sum_0^{\infty} u_m(x)\, u_m(x_0) = \delta(x - x_0)$$

Fourier Transform of Generalized Functions

Lemma: The Fourier transform of a good function is itself a good function

proof: let $g(x)$ be a good function. It's Fourier transform $G(t)$ is given by

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-itx} dx$$

Taking mth derivative wrt t of both sides we get

$$\frac{d^m}{dx^m} G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ix)^m g(x) e^{-itx} dx$$

Integrating by part n- times

$$\frac{d^m}{dx^m} G(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{(-it)^n} \int_{-\infty}^{\infty} \frac{d^n}{dx^n} [(-ix)^m g(x)] e^{-itx} dx$$

$$\left| \frac{d^m}{dx^m} G(t) \right| \leq \frac{1}{\sqrt{2\pi}} \frac{1}{|t|^n} \int_{-\infty}^{\infty} \left| \frac{d^n}{dx^n} x^m g(x) \right| dx$$

Since $g(x)$ is a good function then the integral on the RHS is finite.

Since m is arbitrary $\left|\frac{d^m}{dt^m} G(t)\right|$ goes to zero as $|t| \to \infty$ faster then any power of $\frac{1}{|t|}$

<u>Lemma</u>: Let $g(x)$ be a good function. Then

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{-itx}\, dx$$

implies

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t)\, e^{itx}\, dt$$

<u>proof</u>: Let us introduce an auxiliary function

$$\tilde{g}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t)\, e^{ity - \frac{\varepsilon}{2}t^2}\, dt \quad , \quad \varepsilon > 0$$

which is a good function because $G(t) \cdot e^{-\frac{\varepsilon}{2}t^2}$ is a good function. Let us consider the following difference

$$\tilde{g}(y) - g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t)\, e^{ity - \frac{\varepsilon}{2}t^2}\, dt - g(y)$$

using $G(t)$ given above in the integral

$$\tilde{g}(y) - g(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\,dx \int_{-\infty}^{\infty} e^{-itx + ity - \frac{\varepsilon}{2}t^2}\,dt - g(y).$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\,dx \underbrace{\int_{-\infty}^{\infty} e^{-\frac{\varepsilon}{2}\left[t - \frac{i}{\varepsilon}(x-y)\right]^2 - \frac{(x-y)^2}{2\varepsilon}}\,dt} - g(y)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\,dx \left(\sqrt{\frac{2\pi}{\varepsilon}}\right) e^{-\frac{(x-y)^2}{2\varepsilon}} - g(y)$$

$$= \frac{1}{\sqrt{2\varepsilon\pi}}\int_{-\infty}^{\infty} g(x)\, e^{-\frac{(x-y)^2}{2\varepsilon}}\,dx - g(y)$$

Since
$$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\varepsilon}}\,dx = \sqrt{2\varepsilon\pi}$$

Then

$$\tilde{g}(y) - g(y) = \frac{1}{\sqrt{2\varepsilon\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\varepsilon}}\left(g(x) - g(y)\right)dx$$

$$\Rightarrow \quad |\tilde{g}(y) - g(y)| \le \frac{1}{\sqrt{2\varepsilon\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\varepsilon}}\,|g(x) - g(y)|\,dx$$

$$\le \frac{1}{\sqrt{2\varepsilon\pi}}\left\{\max\left|\frac{dg}{d\xi}\right|\right\}\cdot \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\varepsilon}}\,|x-y|\,dx$$

Let $x - y = du \Rightarrow$

$$|\tilde{g}(y) - g(y)| \le \frac{|\max(\frac{dg}{d\xi})|}{\sqrt{2\varepsilon\pi}} \int_{-\infty}^{\infty} e^{-u^2/2\varepsilon} |u| \, du$$

$$\le \frac{|\max(\frac{dg}{d\xi})|}{\sqrt{2\varepsilon\pi}} \, 2 \int_{0}^{\infty} u e^{-u^2/2\varepsilon} \, du$$

$$\le \frac{2|\max(\frac{dg}{d\xi})}{\sqrt{2\varepsilon\pi}} \left[- e^{-u^2/2\varepsilon} \, \varepsilon \right]_0^{\infty}$$

$$\le \frac{2 \max(\frac{dg}{d\xi})}{\sqrt{2\pi}} \sqrt{\varepsilon}$$

Hence as $\varepsilon \to 0$ $\tilde{g}(y) \to g(y)$ uniformly.

or

$$g(y) = \lim_{\varepsilon \to 0} \tilde{g}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t) e^{-ity} dt.$$

" A useful Integral equality

$$\int_{-\infty}^{\infty} e^{ax - bx^2} dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b} \quad , b > 0$$

Theorem let the sequence $f_n(x)$, $n=1,2,\ldots$ define generalized function $\varphi(x)$. Then the sequence of Fourier ~~series~~ transform of the sequence $f_n(x)$

$$F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_n(x)\, e^{-itx}\, dx$$

defines a generalized function $\Phi(t)$, which is called the Fourier transform of $\varphi(x)$. Furthermore Fourier transform of $F(t)$ is $\varphi(-x)$.

$$\begin{array}{ccc} f_n(x) & \longrightarrow & \varphi(x) \\ \text{FT}\big\downarrow & & \big\downarrow \\ F_n(t) & \longrightarrow & \Phi(t) \end{array}$$

Proof "goodness" of $F_n(t)$ follows from the goodness of $f_n(x)$. Let us show that the integral

$$\int_{-\infty}^{\infty} F_n(t)\, g(t)\, dt$$

exists for any good function $g(t)$ and does not depend on the sequence defining $u(x)$. Let

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x)\, e^{itx}\, dx$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} F_n(t)\, g(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_n(t) \left(\int_{-\infty}^{\infty} G(x)\, e^{itx}\, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x) \left\{ \int_{-\infty}^{\infty} F_n(t)\, e^{itx}\, dt \right.$$

$$= \int_{-\infty}^{\infty} G(x)\, f_n(x)\, dx$$

$$\int_{-\infty}^{\infty} F_n(t)\, g(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{\infty} f_n(x) e^{-itx}\, dx\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) f_n(x) e^{-itx}\, dt\, dx$$

RHS exist hence

$\int_{-\infty}^{\infty} F_n(t)\, g(t)\, dt$ exist and equal

$$= \int_{-\infty}^{\infty} f_n(x)\, G(x)\, dx$$

$$\Rightarrow \underline{\int_{-\infty}^{\infty} \Phi(t)\, g(t)\, dt} = \underline{\int_{-\infty}^{\infty} \varphi(x)\, G(x)\, dx}$$

We write formally

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \varphi(x) e^{-itx}\, dx$$

and $$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(t) e^{itx}\, dt$$

every generalized function has a Fourier transform which is a generalized function and the inversion formula holds without any restriction

Fourier Transform of the δ-function

$$D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \longrightarrow \delta(x).$$

$$D_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx - t^2/4n}\, dt.$$

$$\Rightarrow \quad F_n(t) = \frac{1}{\sqrt{2\pi}}\, e^{-t^2/4n} \qquad n = 1, 2, \ldots$$

proof

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx - t^2/4n}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{4n}(t^2 - 4nix)}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{4n}(t - 2inx)^2 - nx^2}\, dt$$

$$= \frac{1}{2\pi}\, e^{-nx^2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{4n}(t - inx)^2}\, dt}$$

$$a = 0, \quad b = \frac{1}{4n} \qquad \sqrt{\frac{\pi}{b}} = \sqrt{4n\pi}$$

$$= \sqrt{\frac{n}{\pi}}\, e^{-nx^2}.$$

$$\begin{array}{ccc} D_n(x) & \longrightarrow & \delta(x) \\ \downarrow FT & & \downarrow FT \\ F_n(t) & \longrightarrow & \dfrac{1}{\sqrt{2\pi}} \end{array}$$

i.e.,

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{itx}\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx}\, dx$$

FT $\downarrow$

$$\frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x)\, e^{-itx}\, dx = \frac{1}{\sqrt{2\pi}}$$

Fourier transform of $\delta(x)$ is $\dfrac{1}{\sqrt{2\pi}}$

SET 4

MATH 543: GENERALIZED FUNCTIONS

1. A function $g(x)$ that is differentiable everywhere any number of times is called a *good function* if it and its derivatives vanish as $|x| \to \infty$ faster than any power of $\frac{1}{|x|}$. A function $f(x)$ that is differentiable everywhere any number of times is called a *fairly good function* if its modulus and that of its derivatives doesnot increase faster than some power of $|x|$ as $|x| \to \infty$. A sequence of good functions $h_n(x)$ defines a *generalized finction* $\varphi(x)$ through the relation

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(x)\, g(x)dx = \int_{-\infty}^{\infty} \varphi(x)\, g(x)dx$$

where $g(x)$ is any arbitrary good function (test functions). *Regular points* of the generalized functions are those if the sequences defining the generalized functions converge uniformly to an ordinary function in some neighboorhood of these points. The limit of the corresponding sequence at these poins are called the *local values* of the generalized functions. For instance the sequence $h_n(x) = \sqrt{n/\pi}\, e^{-nx^2}$, $(n = 1, 2, \cdots)$ converges uniformly to zero at any point x with $x \neq 0$. Hence the delta function $\delta(x) = 0$ locally for any $x \neq 0$. Prove the following: *If two equivalent sequences (sequences defining the same generlized function) converge uniformly in the neighborhood of $x = x_0$, they determine the same local value of the corresponding generalized function.*

2. Let $a_n(x), b_n(x)$, $(n = 1, 2, \cdots)$ be sequences of good functions defining , respectively the generalized functions $\alpha(x), \beta(x)$. Let $g(x)$ be a good function and $f(x)$ be a fairly good function. Prove the following:
(i) $p\, a_n(x) + q\, b_n(x)$ defines the generalized function $p\alpha(x) + q\beta(x)$ where p and q are arbitrary real numbers.
(ii) $f(x)\, a_n(x)$ defines the generalized function $f(x)\alpha(x)$
(iii) $\frac{da_n(x)}{dx}$ defines the generalized function $\frac{d\alpha(x)}{dx}$.

(iv) Give an example where two sequences $a_n(x)$ and $b_n(x)$ define the generalized functions $\alpha(x)$ and $\beta(x)$ respectively but their product doesnot converge to a generalized function in the following sense

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} a_n(x)b_n(x)g(x)dx$$

Consider for instance the product sequences in problem 4.

3. Prove the following for the Dirac δ-function:
(i) $\int_a^b \delta(x-c)\, f(x)dx = f(c) \quad$ if $a \leq c \leq b \quad$ otherwise zero.
(ii) $\delta(ax) = \frac{1}{|a|}\,\delta(x)$
(iii) $f(x)\delta(x) = f(0)\,\delta(x)$
(iv) $\delta(f(x)) = \sum_{n=1}^{N} \frac{\delta(x-x_i)}{|df/dx|_{x=x_i}}$, where $f(x)$ has $N-$ roots x_i, $\quad i = 1, 2, \cdots N$
(v) $f(x)\delta'(x) = -f'(0)\,\delta(x)$

4. Prove that each of the following sequences define the Dirac δ-function: (For all cases $n = 1, 2, \cdots$)
(i) $D_n^1 = \frac{n}{\sqrt{\pi}}\, e^{-n^2 x^2}$
(ii) $D_n^2 = \frac{1-\cos nx}{\pi n x^2}$
(iii) $D_n^3 = \frac{n}{\pi}\frac{1}{1+n^2x^2}$
(iv) $D_n^4 = \frac{\sin nx}{\pi x}$.

Prove the following:
(a) $\int_{-\infty}^{\infty} D_n^i(x)dx = 1$, $i = 1, 2, 3, 4$ for all n
(b)$\lim_{n\to\infty}\int_{-\infty}^{\infty} D_n^i(x) f(x)dx = \int_{-\infty}^{\infty} \delta(x)\, f(x)dx = f(0)$, $i = 1, 2, 3, 4$.

5. Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}$$

(i) Prove that $h_n(x) \to \theta(x)$ where $\theta(x)$ is the step function and
(ii) $\frac{dh_n(x)}{dx} \to \delta(x)$. Hence formally we may write that $\frac{d\theta(x)}{dx} = \delta(x)$

6. Prove the following:

(i) Let $f(x)$ be a good function then its Fourier transform is also a good function.

(i) Let $f(x)$ be a good function. Then

$$G(t) = \frac{1}{sqrt2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-itx}\, dx$$

implies

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(t)\, e^{itx} dx$$

(iii) Let the sequence $f_n(x)$ $(n = 1, 2, \cdots)$ defone the generalized function $\varphi(x)$. Then the sequence of of Fourier transforms of the sequence $f_n(x)$

$$F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_n(x)\, e^{-itx}\, dx$$

defines a generalized function $\Phi(t)$, which is called the Fourier transform of $\varphi(x)$.

7. (a) Let $u_m(x)$, $(m = 0, 1, 2, \cdots)$ be one of the classical orthonormal polynomials with weight function $w(x)$ and $x \in [a, b]$. The the sequence

$$h_n(x) = w(x) \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [a, b]$ defines the delta function $\delta(x - x_0)$

(b) Let $u_n(x) = \frac{1}{\sqrt{2\pi}} e^{-inx}$, $(n = 0, 1, 2, \cdots)$ with $x \in [-\pi, \pi]$. Prove that the sequence

$$h_n(x) = \sum_{k=0}^{n} u_k(x)\, u_k(x_0)$$

where $x_0 \in [-\pi, \pi]$ defines also the delta function $\delta(x - x_0)$

You first prove that $\int_I h_n(x) dx = 1$ for all n, where I is the corresponding interval. Then prove that $\lim_{n\to\infty} \int_I f(x)\, h_n(x) dx = f(x_0)$

8. Higher dimensional delta functions can be defined. For instantce in three dimensions the delta function $\delta^3(\mathbf{x})$ can be represented in terms one

dimensional delta functions. The way we achieve this is to use the general identity $\int_V \delta^3(\mathrm{x})d^3x = 1$. Hence using this property prove that

(a) In Cartesian coordinates $\delta^3(\mathrm{x}) = \delta(x)\,\delta(y)\,\delta(z)$

(b) In spherical coordinates $\delta^3(\mathrm{x}) = \frac{1}{r^2 \sin\theta}\,\delta(r)\,\delta(\theta)\,\delta(\varphi)$

(c) In cylindrical coordinates $\delta^3(\mathrm{x}) = \frac{1}{\rho}\,\delta(\rho)\,\delta(z)\,\delta(\varphi)$

(d) In three dimensions verify the following identity and find the constant α_3 in the following equation

$$\nabla^2 \frac{1}{|\mathrm{x}|} = \alpha_3\, \delta^3(\mathrm{x})$$

where ∇^2 is the three dimensional Laplace operator (Laplacian).

(e) The above Laplace equation can be written for in any dimension n

$$\nabla^2 \frac{1}{|\mathrm{x}|^{n-2}} = \alpha_n\, \delta^n(\mathrm{x})$$

where $\delta^n(\mathrm{x})$ is the n dimensional δ function and α_n are just constants. Here $n \neq 2$. For $n = 2$ we have

$$\nabla^2 \log|\mathrm{x}| = \alpha_2\, \delta^2(\mathrm{x})$$

Find $\alpha_n, \quad n \neq 2$ and α_2